

THE
THEORY
OF THE
MOTION
OF THE
APSIDES

, In GENERAL,
AND OF THE
APSIDES of the Moon's Orbit
In PARTICULAR.

Written in FRENCH,
By D. C. WALMESLEY, B. A.

And now carefully Revised and Translated into
ENGLISH.

LONDON:
Printed for W. OWEN, at *Homer's Head*, in
Fleet-street. MDCCCLIV.

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ADVERTISEMENT.

THE following Performance, containing many geometrical Demonstrations, which may be of great Use in perfecting the Theory of the Moon, it was thought a Translation would prove acceptable to the *English* Astronomers and Geometricians.

In the Course of this Work, the ingenious Author has clearly exposed the Fallacy of a plausible Objection which *M. Clairaut* formerly made to the grand Principle of the *Newtonian* Philosophy. The Objection was produced by a Deduction from Geometry, and therefore its Truth or Falsity was readily to be discovered by that Science and just Observations.

The Property of Attraction is demonstrated by a thousand Experiments, and its Laws are established upon indubitable geometrical Principles. Our Ignorance of the Cause of Attraction can no more render the Property itself precarious, than we can doubt, that the Con-

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traction of the Muscles is really the Cause of Motion in Animals, although the Cause of muscular Contraction remains undiscovered.

It ought to be observed, that instead of the differential Calculus made use of by the Foreigners, the Method of Fluxions is here used as more agreeable to the *English* Reader.

All I shall add further, is this; that all the mathematical Calculations have been strictly examined and carefully corrected by Mr. EMERSON. And whatever Errors have escaped in the Original, are rectified in this Translation; which I presume will render this Work the more acceptable to the curious Reader.

J. BROWN.

THE



T H E
P R E F A C E.



THE Philosophy of Sir *Isaac Newton*, seemed to have attained such a Degree of Certainty, by its Conformity with all the Appearances of Nature, that there remained no Room to doubt of its being established on incontestable Principles. The scrupulous Exactness which Geometricians have observed in their Enquiries, served only to confirm the System of this great Man, till *M. Chairaut*, of the Academy of Sciences, whose Merit is well known among Geometricians, in a Memoir which he read to the Academy in 1747, pretended to find by his Calculations, that the universal Law of Gravitation, established by Sir *Isaac Newton*,
namely,

namely, that its Force is in an inverse Ratio of the Square of the Distance, will give to the *Moon's Apogee* only one half of the Motion which Astronomers have discovered by their Observations; from whence he concludes, that it will be necessary to change this Law, by adding *Something* to correct its Insufficiency.

It is astonishing, that a Principle founded on so wonderful an Agreement with all the other Appearances, should fail in this single Case; and still more, since, if this Discovery take Place, we cannot help accusing Sir *Isaac Newton*, and Mr. *Machin*, of an Error too gross to be imputed to such great Men; for both these Geometricians have calculated the Motion of the *Moon's Apogee*; and altho' they have suppressed their Calculations, yet the Result which they have given us agrees with Observations: How comes it then, that they have drawn, from a Principle well known to them, Conclusions which ought not to follow? Sir *Isaac* having demonstrated, that not only all the celestial Motions in general are regulated by this Law of the inverse Ratio of the Square of the Distance, but that even all the different Changes which happen in these Motions

tions are only necessary Consequences of this same Law, determined the precise Quantity of all the Varieties which are found in the Motion of the Moon; and Astronomers, after remarking these irregular Phenomena, acknowledge at present the most exact Conformity between Sir *Isaac Newton's* Theory and their Observations: If therefore such a Number of Calculations, which all agree with Experience, be contradicted by one only, can we avoid concluding that it must be falsely deduced? Or is it difficult to arrive at the Truth, by a Road which leads us another Way.

These Reasons appeared sufficient to afford Room for suspecting some Fault in *M. Clairaut's* Method. That learned Academician having communicated his Memoir to me after it was printed, I found that his Theory was built upon a Hypothesis, which would no way agree with the Question to which he applied it. From the general Solution which he gives of the famous Problem of *three Bodies*, he draws an Equation to represent that of the Moon's Orbit; but as this Equation is found too general for the Use which he would make of it, he is obliged to have

have Recourse to a particular Supposition, to wit, that the Motion of the Moon may be exactly enough represented by a Curve, formed by the Motion of an Ellipsis around its Focus; afterwards, by comparing his general Equation with the Equation of this Curve, he endeavours to determine the Value of the Quantities which he has Occasion to know. But after the Introduction of this Hypothesis, it is plain that the preceding Calculation became useless, and it would have been more simple, to have deduced the Motion in Question immediately from the Hypothesis: Indeed we shall find, that the Result would be precisely the same, as that which *M. Clairaut* has drawn from the two combined Equations.

I represented to *M. Clairaut*, in December, 1748, some of the principal Reasons which induced me to think this Hypothesis incompatible with the Motion of the Moon, especially with that of her Apogee; these Reasons are in my first Chapter. When we would make use of any arbitrary Hypothesis, we cannot be too circumspect; for if we be not well assured, that they will agree with the Nature of the Thing to which they are applied,

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plied, it frequently happens, that they draw us into Consequences irreconcilable to what has already been demonstrated by Experience, and by direct and real Methods. Thus the Nature of this Hypothesis, which we are now treating of, has obliged *M. Clairaut* to admit as Truths, *That the Motion of the Apogee of the Moon is made in an uniform Progression, and that the Excentricity of its Orbit is not subject to any other Variation*; two Things which equally contradict both the Astronomers and Geometricians. These Reflections are sufficient to make us conclude, that this Theory ought to be looked on as defective, and that there is no Necessity of changing the universal Law of Gravitation before laid down, by substituting a new Expression: To this may be added, the great Inconveniencies that would result to Philosophy from such a Change, which have been more than sufficiently demonstrated by an illustrious Member * of the same Academy.

To consider the Thing more narrowly, having applied myself to discover by direct Methods the Motion of the Apfides, I found

* *M. de Buffon.*

two general Methods, which I've applied to the Case of the Moon, and which have given me the Motion of its Apogee, entirely agreeable to Observations ; that is to say, something more than three Degrees during each Revolution of the Motion : In these Methods, to free the Question as much as possible from Difficulty, I have resolved the Force of the Sun upon the Moon into two Parts, and have only employed that Part which increases or diminishes the Gravity of the Moon towards the Earth, which, according to Sir *Isaac Newton's* Demonstration, is the Cause of the Motion of the Apogee ; this Manner of resolving the Problem, appeared to me the most simple, and least subject to Error ; I have supposed, that the Orbits are a little excentrical, which *M. Clairaut*, and the greatest Part of the Geometricians, acknowledge, will occasion no sensible Mistake in the Determination of the Motion of the Moon's Apogee. There is, moreover, a third particular Method for calculating the Motion of this Planet ; it depends on a Proposition, whose Demonstration I have not given, but I thought we might depend on the Authority of Sir *Isaac Newton*.

Having

Having thus engaged in this Question, I thought it would prove of Service to the Sciences, if I communicated some Part of my Researches to the Public; but, whilst I was preparing this Piece, I learnt that *M. Clairaut*, having discovered the Fault which had slipped into his Theory, had changed his Opinion, and was endeavouring to find some Means to make it quadrate with Observations: Thus the Principle of Gravitation is re-established completely, and the Conclusions which I have drawn from my Method are confirmed, which has determined me to publish them.

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CHAP. I.

SECT. I.



THE celebrated Academician, after having formed the general Equation, $\frac{p}{r} = 1 - c \text{ Cof. } U + \text{Sin. } U \int \Omega \text{ Cof. } U dU - \text{Cof. } U \int \Omega \text{ Sin. } U dU$, of the Orbit described by the Moon, according to Sir *Isaac Newton's* Law, in order to determine the Motion of its

2 *The* THEORY of

its Apogee, introduces a new Equation, namely,
 $\frac{k}{r} = 1 - e \text{ Cof. } m U$, which is that of a Curve
described by uniformly moving an Ellipsis around
its Focus, in such a Manner, that its Apside
shall describe an Angle, which shall be to that
which the Moon runs over in this Ellipsis, as
 $1 - m$ to 1 ; by the Help of this particular Equa-
tion, he makes Substitutions in the general Equa-
tion, and afterwards by equalling the Terms of
the two Equations, he deduces a Form by which
he finds that the Motion of the Apogee of the
Moon ought to be about 20 Degrees in a Year;
that is to say, one Half of the Motion which
it really makes.

This Process is not right for two Reasons.

S E C T. II.

THE first is this. — If the Equation $\frac{k}{r}$
 $= 1 - e \text{ Cof. } m U$, which is supposed
very nearly to represent that of the Orbit of the
Moon, does not truly represent it, but is even
greatly different from it, then the Conclusion,
which he deduces by reducing the general Equa-
tion to it, cannot be conformable to the Truth:
But it is certain that this Equation cannot agree
with

with the Moon's Orbit, since it supposes that the Apogee of the Moon always advances, and this Advancement is made by a constantly uniform Motion; whereas the real Apogee sometimes recedes, sometimes advances, and that too by a Motion which is not uniform; this is very plain, because the Difference of the Forces which the Sun exerts upon the Earth and upon the Moon is, in Proportion to the Force of the Earth upon the Moon, sometimes additive, sometimes subtractive, according to the Angle of the Elongation of the Moon from the Sun: When it is additive, that is to say, when it acts in the same Direction with that of the Earth, which happens about the *Quadratures*, then the two conjoined Forces form one compound Force, which is in a Ratio *less* than the reciprocal one of the Square of the Distance from the Center of the Earth; this makes the Apogee move retrograde, as it constantly is, by what Sir *Isaac Newton* has demonstrated, and by what you'll find in the next Chapter: But when this Difference is subtractive, that is, when it acts in a Direction contrary to that of the Earth, which happens towards the *Syziges*, then the Difference of the two Forces in Question encreases in a Ratio *greater* than the reciprocal one of the Square of the Distance; which makes the Apogee advance: Consequently,
the

4 *The* THEORY of

the Hypothesis of the uniform Progression of the Apogee cannot be admitted.

S E C T. III.

OUR second Reason against this Process is, that the Hypothesis of a moveable Ellipsis is only purely mathematical, and hath no Existence in Nature; consequently, the Conclusions drawn therefrom will widely mislead us from what we learn by Observations: For Sir *Isaac Newton* *, having expressly examined this Hypothesis, demonstrates, that in the Case of the Moon it would only give one Half of the Motion of the Apogee, and thus agrees with *M. Clairault* in this Point: But Sir *Isaac*, suspecting the Defect of the hypothetic Method, abandoned every Supposition, and sought, by a direct Method, to find what ought to be the Quantity of the Motion of the Apogee, (as may be seen in the Scholium inserted at the End of his Theory of the Moon, in the third Book of the *Principia*, Ed. I.) And he found, that this Motion was about 40 Degrees annually—

* Princip. Math. Lib. I, Sect. IX.

the MOTION of the APSIDES, &c. 3

Similibus computationibus, says he, *inveni quod ipsius (Apogæi) motus medius annus sit quasi grad. 40.* His Calculation then agrees here with the Observations.

Mr. *Machin* likewise having sought, by a particular Method, to know what the same Supposition of a moveable Orbit would produce, at length found * it only about one half of the Apogee, as Mr. *Clairaut* and Sir *Isaac Newton* have both done; and for this Reason he contrived a direct Method, by which he found the annual Motion of the Apogee so conformable to astronomical Observations, that the Difference was only four Seconds: We see then that Sir *Isaac Newton* and Mr. *Machin* both gained the same End, by Methods vastly different from each other. As to the moveable Orbit, their Calculation agrees with that of M. *Clairaut*, but by directly following Nature, it is found to agree with Observations; this may convince us, that the Hypothesis can only be adopted as purely mathematical.

* The Laws of the Moon's Motion, p. 30, 31, &c.

C

S E C T.

S E C T. IV.

IT will not appear suprizing, that this Hypothesis of the moveable Orbit does not agree with the real Motion of the Moon, if we consider these two Things.

1. That the Difference of the Forces required for a Body to describe a fixed Ellipsis and a moveable Ellipsis, is, in an inverse Ratio of the Cube of the Distance, as Geometricians well know; but the *disturbing Force* of the Sun, which will be the Difference of the Forces, in this Hypothesis, does not at all agree to this Proportion.

2. It is a geometrical Truth, that the Encrease or Diminution of the central Force does not change the Quantity of the Area described in a given Time, by a body which makes its Revolution round the Centre; whence it follows, that the Moon actually describes the same Area round the Earth in a given Time, as if the Action of the Sun was nothing, regarding only that Part of the solar Action which is directed to the Centre of the Earth, and which gives Motion
to

to the Apogee; but this Law cannot take Place into the moveable Ellipsis, because, by rendering the Ellipsis moveable, the Area described is not the same in a given Time, as in the immoveable Ellipsis.

S E C T. V.

BUT it may perhaps be objected, that Sir *Isaac Newton*, in calculating the Motion of the Apside, during one Revolution of the Moon, in a moveable Orbit, employs only half of the action of the Sun, as *M. Clairaut* remarks; and thus, by taking the whole Force, he would have found the whole Motion: And in this Case the Hypothesis of the moveable Orbit may be admitted, even according to Sir *Isaac* himself.

To this it may be answered, that Sir *Isaac Newton* indeed only employs one half of the solar Force in his Calculation; for this Force is the 178,725th Part of the Moon's Gravity towards the Earth, and Sir *Isaac* only takes the 357,45th Part of this Gravity: But we must remember, that he supposes the Force, expressed by $\frac{1}{357.45}$, acts during the whole Revolution of the Moon;

and it is evident, that in order to find the Motion of the Apogee in this Supposition, he ought only to take this Part, viz. $\frac{1}{557.45}$ of the total Force ;

For since, according to the Principles of this great Man, the Motion of the Apogee is really retrograde during half of the Moon's Revolution, it is plain, that to find the Quantity of the direct Motion of the Apogee, it is indifferent whether we take the entire Force, supposing it to act during the Time of half a Revolution; or half the Force, supposing that half to act during the whole Revolution: It is then certain, that Sir Isaac Newton has found the whole Motion of the Apogee, as it must result from the Hypothesis of the moveable Ellipsis.

S E C T. VI.

P E R H A P S it will be further objected, that as the three first Terms only of the general Equation are compared with those of the Equation; $\frac{k}{r} = 1 - e \text{ Cos. } m U$ of the moveable Ellipsis, in order to have the Value of the three indeterminate Quantities, k , e , m , the Terms which remain in the general Equation, after this Reduction

Reduction, would have a considerable Value, if the moveable Ellipsis does not correspond to the Moon's true Orbit; for if they do correspond, the two Equations must really agree, as the Terms which remain in the general Equation would be equal to 0, or at least to a very small Value: But it happens, that, by using this Equation, the Terms which remain after the Reduction, have only a very small Value; hence we ought to conclude, that the Motion of a moveable Ellipsis is very nearly correspondent to the true Motion of the Moon.

But this Objection will have no Weight in the present Question, if we consider that the Reduction of the general Equation makes the two Equations necessarily agree; for the two last Terms of the general Equation have a very small Value, because Ω , which multiplies them, is a very small Quantity, as the Theory informs us, and as *M. Clairaut* himself acknowledges; if we equal the first Terms of the general Equation to the Terms $1 - e \text{ Cos. } m U$, it is plain that the other Terms which remain, and which are multiplied by the Quantity Ω , must necessarily be very small, and not worth Notice. And from this it will also follow, that, as long as the Value of Ω remains inconsiderable, we may introduce other Equations, quite different from that, which agrees

agrees to the Moon's Orbit, to which nevertheless the general Equation, after being reduced, will agree; or the difference will not be material.

S E C T. VII.

LASTLY, to prevent as much as possible, every Objection which can be made, I think no one can doubt, that the Method of this learned Academician cannot be founded on the Hypothesis of a moveable Ellipsis; this is clear by the Reduction he makes, which necessarily gives to the general Equation the same Form with that of the moveable Ellipsis, as we have seen: But to be still more assured, we need only compare the Formula, which *M. Clairaut* draws from his Equation, to determine the Motion of the Apogee, with that which *Sir Isaac Newton* found by his Method in the same Hypothesis. Thus taking the Formula of the first, $m^2 = 1 - \frac{3a}{2}$, in

which we have $a = \frac{1}{178.725}$, or $a = 2c$,

according to *Sir Isaac Newton*, we shall have

$m^2 = 1 - 3c$, which gives $m = 1 - \frac{3c}{2}$, very

nearly, c being a very small Quantity; and since

the MOTION of the APSIDES, &c. 11

since the Angle described by the moveable Orbit of the Moon, during one Revolution, is to 360 Degrees, as $1 - m$ to 1; by calling this Angle z , we shall have $360^\circ + z = 360^\circ \times 1 + \frac{3c}{2}$, which is the Angle described by the Moon in the moveable Ellipsis, during her Revolution from one Apogee to the other.

The Formula of Sir *Isaac Newton* to find the same Angle, is $360^\circ \sqrt{\frac{1-c}{1-4c}}$; but $\frac{1-c}{1-4c} = 1 + 3c$ very nearly; then $\sqrt{\frac{1-c}{1-4c}} = 1 + \frac{3c}{2}$ and $360^\circ \sqrt{\frac{1-c}{1-4c}} = 360^\circ \times 1 + \frac{3c}{2}$. This proves that the Formulæ of these two Geometricians are the same; and so the Orbit expressed by the general Equation, when reduced, is precisely that of the moveable Ellipsis. But I think the Reasons before alledged are sufficient to prove that this Hypothesis is defective, and that we ought rather to hold to the direct Calculations of Sir *Isaac Newton*, and Mr. *Machin*, which depend on no Hypothesis, and whose Results agree with Observations.

C H A P.



C H A P. II.

General Methods of finding the Motion of the Apfides in Orbits a little excentrical, and particularly that of the Apfides of the Moon's Orbit.

M E T H O D I.

S E C T. VIII.

THIS first Method consists in finding the Time which a Body ought to take up in descending in a right Line from the Point of the superior Apfide to the nearest Point of the Centre to which the Body can arrive, which is the inferior Apfide, considering only its rectilineal Motion, as it approaches the Centre, without paying any Regard to the circular Motion; this Time, as it is plain, is equal to that of the Revolution of one Apfide to another. Let $A P \bar{D}$ (Fig. 1.) be the Curve described in virtue of a Force which tends to the Point T , and which let be as the Power n of the Distance from the Centre;

Centre; draw from the Centre T to the Curve any line TP , and another Tp infinitely near to TP , and from the same Centre draw the Arches of a Circle PQ , pq , to meet the Line of the Ap-sides TA in Q and q , and let m be the Interfection of the Lines PT , pq ; it is evident that the Motion Pp is compounded of the circular Motion pm , and the rectilineal Motion Pm , by which the Body approaches to the Centre: Calling then TA , 1 ; TP , x ; and the Velocity with which the Body describes Pm or Qq , v ; and supposing that the centripetal Force at the superior Ap-side A is $= 1$; and the centrifugal Force $= q$, which varies, as is known, reciprocally, as the Cube of the Distance, it is plain that the Force which makes the Body descend towards the Centre, is the Difference between the centripetal and the centrifugal Force, that is to say, this Force is

$x^n - \frac{q}{x^3}$; consequently, according to the Prin-ciples of accelerating Forces, we shall have $\dot{vv} = -x^n \dot{x} + \frac{q\dot{x}}{x^3}$, which gives $\frac{vv}{2} = -\frac{x^{n+1}}{n+1} -$

$\frac{q}{2x^2} + \frac{1}{n+1} + \frac{q}{2}$; x being $= 1$ at the Time

that $v = 0$; from hence, by putting t for the Time of the Descent through Pm , and neglecting the constant Quantities which multi-ply the Formula, because we are searching the

D

Ratios,

Ratios, and not the Equalities, we shall have $i =$

$$\frac{-\dot{x}}{\sqrt{\frac{-x^{n+1}}{n+1} - \frac{q}{2x^2} + \frac{1}{n+1} + \frac{q}{2}}}, \text{ or } i =$$

$$\frac{-\dot{x}}{\sqrt{-2x^{n+3} + n+1 \times qx^2 + 2x^2 - n+1 \times q}}$$

Let $x = 1 - z$, and we shall have $x^{n+3} = 1$

$$- n+3 \times z + n+3 \times \frac{n+2}{2} \times zz + \text{Ec. and}$$

the Trajectory being, by the Hypothesis, very little different from a Circle, we may neglect all the Terms in which z arises to more than two Dimensions; and therefore we shall have $i =$

$$\frac{1 - z \times \dot{z}}{\sqrt{2 - 2q \times z - n+4 - q \times zz}}, \text{ or more simply,}$$

$$i = \frac{1 - z \times \dot{z}}{\sqrt{\frac{2 - 2q}{n+4-q} z - zz}}.$$

Put $\sqrt{\frac{2 - 2q}{n+4-q} z - zz} = 0$, and we shall

find $z = 0$, and $z = \frac{2 - 2q}{n+4-q}$; these two

Values of z determine the two Points where the Velocity v is nothing; that is to say, if we

take $AB = \frac{2 - 2q}{n+4-q} \times TA$, B is the Point nearest the Centre to which the Body can arrive.

S E C T.

S E C T. IX.

NOW upon the Diameter AB describing the Semi-circumference ACB, and drawing the Ordinate QC, the Fluent of

$$\frac{1 - z \times \dot{z}}{\sqrt{\frac{2 - 2q}{n+4-q} z - zz}}$$

will be $\frac{n+4-q}{1-q} \times AC$

$$+ CQ - AC = \frac{n+3}{1-q} \times AC + CQ, \text{ then the}$$

Time t of the Descent through AQ will be proportional to this Quantity. If we take the infinitely small part AN, and raise the Perpendicular NRM, which shall meet the Circumference ACB at R, and the Arch of the Circle AM, described from the Centre T at M, then we shall have NR = AR, and consequently the Time of the Descent through AN, proportional to $\frac{n+4-q}{1-q} \times AR$; but this Time is to the

Time that the Body would take up in descending through AN, by Virtue of the centripetal Force alone, or, which comes to the same, in describing the Arch AM, as 1 is to $\sqrt{1-q}$; because, the Spaces being equal, the Times are in an inverse Ratio of the Square-Roots of the accelerating Forces; and the Time of describing the

Arch AM, is to the Time of half a Revolution in the Circle AM, as the Arch AM is to the Semi-circumference; from hence, by Composition of all these Ratios, one may find the Ratio of the Time of this Half-Revolution, to the Time of the Descent through AB: But, in order to put these Ratios and their Result more distinctly before the Eye, let us suppose in general, that some such Expression as T.AB denotes the Time of the Descent through AB, &c. and, according to the foregoing Reasoning, we shall have T.AB : T.AN :: $n+3 \times ACB$: $n+4-q \times AR$. T.AN : T.AM :: 1 : $\sqrt{1-q}$. T.AM : R :: AM : C. (Where R denotes the Time of half the Revolution in the Circle AM, and C the Semi-circumference of the Circle.) From hence, by the Composition of these Ratios, T.AB : R :: $n+3 \times AGB \times AM$: $n+4-q \sqrt{1-q} \times AR \times C$.

But we shall also have,

$$ACB : C :: 1-q : n+4-q;$$

$$AM : AR :: \sqrt{n+4-q} : \sqrt{1-q}.$$

From whence we shall find,

$$T.AB : R :: n+3 : n+4-q^{\frac{1}{2}}.$$

But since we suppose the Orbit to be very nearly circular, we shall have $q=1$ very nearly,

and

and consequently, $T.AB : R :: 1 : \sqrt{n+3}$.
Then, by substituting the Angles, instead of
the Times, the Angle comprised between the two
Apfides will be equal to $\frac{180^\circ}{\sqrt{n+3}}$, which is
conformable to that which we find in the Hypo-
thesis of a moveable Orbit.

S E C T. X.

NOW, as we would principally treat of the
Motion of the Moon's Apogee, let us sup-
pose that the centripetal Force is in an inverse Ratio
of the Square of the Distance from the Centre,
and that we add to it a new Force, extremely
small, which varies in a direct Ratio of the
Distance, and which may be to the former, at the
Distance TA, as ϕ to 1; in this Case we shall
have $v\dot{v} = -\frac{\dot{x}}{x^2} - \phi x\dot{x} + \frac{q\dot{x}}{x^3}$, which gives

$$\frac{v\dot{v}}{2} = \frac{1}{x} + \frac{\phi}{2} \times \frac{1}{1-x^2} - \frac{q}{2x^2} - 1 + \frac{q}{2} :$$

From whence, by neglecting the constant Quan-
tities, we shall have

$$\dot{i} = \frac{-x\dot{x}}{\sqrt{-q + 2x - 2x^2 + qx^2 + \phi x^2 - \phi x^4}} .$$

Afterwards supposing $1 - z = x$, and rejecting
the

18 *The* THEORY of

the Terms in which z hath more than two Dimensions, we shall find $t = \frac{1 - z \times \dot{z}}{\sqrt{\frac{2 - 2q + 2\phi}{2 - q + 5\phi}} z - z\dot{z}}$:

From which it is plain, that by taking (Fig. 1.)

$AB = \frac{2 - 2q + 2\phi}{2 - q + 5\phi} \times TA$, and describing upon this Diameter the Semi-circumference ACB ,

we shall have $t = \frac{1 + 4\phi}{1 - q + \phi} \times AC + CQ$;

consequently, if we suppose that the Circle AM is the Orbit that the Moon would describe, by virtue of the Action of the Earth alone; by a like Reasoning with that in the preceding Section; we shall have,

$$T.AB : T.AN :: 1 + 4\phi \times ACB : 2 - q + 5\phi \times AR.$$

$$T.AN : T.AM :: 1 : \sqrt{1 - q + \phi}.$$

$$T.AM : R :: AM : C.$$

And, by the Composition of these Ratios, $T.AB : R :: 1 + 4\phi \times ACB \times AM : 2 - q + 5\phi \sqrt{1 - q + \phi} \times AR \times C$.

But because we have,

$$ACB : C :: 1 - q + \phi : 2 - q + 5\phi;$$

$$\text{and } AM : AR :: \sqrt{2 - q + 5\phi} : \sqrt{1 - q + \phi};$$

It becomes,

$$T.AB : R :: 1 + 4\phi : 2 - q + 5\phi^{\frac{3}{2}}.$$

But

But since, by Supposition, the Force ϕ is very small, and therefore the Orbit very near circular, upon Supposition that $q = 1$; and since the real Force, by whose Power the Moon describes her Orbit, is not only the Gravity caused by the Action of the Earth, but the Sum of that Gravity, and the Action ϕ of the Sun; it follows, that the Semi-Revolution R is to the real Semi-Revolution of the Moon, which I call L, in the Ratio of $\sqrt{1+\phi}$ to 1; which will give

$$T.AB : L :: \overline{1 + 4\phi\sqrt{1+\phi}} : \overline{1 + 5\phi}^{\frac{3}{2}}.$$

But $\frac{\overline{1 + 5\phi}^{\frac{3}{2}}}{1 + 4\phi} = \sqrt{1 + 7\phi}$. Therefore if, instead of the Times, we substitute the mean Motions; the Angle described by the Moon, in its Revolution, from one Apogee to the other, will be equal to $360^\circ \sqrt{\frac{1+\phi}{1+7\phi}}$, or rather, because the Force ϕ is negative with respect to the Moon, $360 \sqrt{\frac{1-\phi}{1-7\phi}}$, and that Force ϕ , as is known, is the 357.45th Part of the Gravity of the Moon towards the Earth; therefore making $\phi = \frac{1}{357.45}$, the Angle between the two Apogees will be $360 \sqrt{\frac{356.45}{350.45}} = 363^\circ 4' 7''$; which
causes

causes the Apogee to advance $3^{\circ} 4' 7''$ in one Revolution; which is exactly conformable to astronomical Observations.

S E C T. XI.

FROM the Formula which we have used, to find the Motion of the Apogee of the Moon, this Proportion may be evidently deduced: *Viz. The mean Motion of the Moon from the Apogee, is to the mean Motion of the Moon; as $\sqrt{1-7\phi}$ is to $\sqrt{1-\phi}$* : We shall find this Proportion is the same which * Mr. Machin gives us, without any Demonstration, if we diligently remark its Construction.

After this Manner I have endeavoured, as far as I was able, to unravel and analyse the Idea which Mr. Machin gives us in Page 64. of the Treatise I mention, when he says, “*That there*
 “*is a general Method to determine the Motion of*
 “*a Body, as it approaches to, or recedes from,*
 “*the Centre of Force; without considering its Mo-*
 “*tion round that Centre; and that this Method*

* The Laws of the Moon's Motion, P. 32.

“ seems

*“ seems to be necessary for knowing the Laws of
“ the Revolution of a Body, with Respect to the
“ Apfides of its Orbit, &c.”* It is to be wished,
that this learned Geometrician had himself pub-
lished his Method; it would undoubtedly have
been much more perfect than this which is here
given.

METHOD II.

SECT. XII.

IN the preceding Method, we have set aside
one Part of the Motion of the Body, in search-
ing for the Motion of the Apfides; here we shall
follow a different method, and consider the whole
Motion of the body in its Orbit, without any Re-
solution of it: In order to do this, we must first
give a Solution of the Problem contained in the
following Lemma.

LEMMA.

L E M M A.

To find the Height from which a Body must fall, in order to acquire the Velocity with which it describes a Circle at any Distance from the Centre.

Let C (Fig. 2.) be the Centre of Force, and E the Point from whence the Body must fall, to acquire at P the Velocity with which it is supposed to describe a Circle, of which CP is the Radius; it is demonstrated (Prop. 39. Lib. 1. *Princ. Math. Newton.*) that if the central Force is in the Ratio of PC^n , the Velocity acquired by the Body at P, in falling from the Point E, is in

the Ratio of $\sqrt{CE^{n+1} - CP^{n+1}}$; and we know that the Velocity in the Circle is as the Square-Root of the Product of the Force by the Distance to the Centre, that is to say, as

$CP^{\frac{n+1}{2}}$: Taking then the infinitely small Part EM, and calling the Velocity u , which the Body has acquired at P, by falling from the Point E, and C the Velocity in a Circle whose Radius is CP, v the Velocity acquired by falling through the small Space EM, and c the Velocity in a Circle, whose Radius is CE, we shall have the following

following Proportions (observing that $\overline{CE}^{n+1} = \overline{CM}^{n+1} = n+1 \times \overline{CE}^n \times \overline{EM}$):

$$\overline{UU} : \overline{vv} :: \overline{CE}^n - \overline{CP}^n : n+1 \times \overline{CE}^n \times \overline{EM}.$$

$$\overline{uu} : \overline{cc} :: \frac{2\overline{EM}}{\overline{CE}^{n+1}} : \frac{\overline{CE}^n}{\overline{CP}^{n+1}}.$$

From whence we draw,

$$\overline{UU} : \overline{CC} :: \overline{CE}^{n+1} - \overline{CP}^{n+1} : \frac{n+1}{2} \overline{CP}^{n+1}.$$

Making then $\overline{UU} = \overline{CC}$, we shall have $\overline{CE}^{n+1} =$

$$\frac{n+3}{2} \overline{CP}^{n+1}; \text{ and if } \overline{CP} = 1, \text{ it follows that}$$

$$\overline{CE}^{n+1} = \frac{n+3}{2}.$$

SECT. XIII

BUT if the central Force be composed of the Sum of two Forces, whereof the one acts in the Ratio of $\overline{CP}^n$, and the other in the Ratio of $\overline{CP}^m$, we shall suppose that the whole Force is expressed in this Manner, $\overline{CP}^n + a \overline{CP}^m$, a being a constant Quantity; then by applying to this Case the Demonstration of the 39th Proposition of

Sir *Isaac Newton* above cited, we shall find, that the Square of the Velocity acquired in falling from the Height EP , is proportional to $\frac{1}{n+1} \times$

$$\overline{CE}^{n+1} - \overline{CP}^{n+1} + \frac{2\phi}{m+1} \times \overline{CE}^{m+1} - \overline{CP}^{m+1},$$

from whence, by the same Reasoning as in the preceeding Case, we deduce $UU : CC :: \frac{2}{n+1}$

$$\times \overline{CE}^{n+1} - \overline{CP}^{n+1} + \frac{2\phi}{m+1} \times \overline{CE}^{m+1} - \overline{CP}^{m+1}$$

: $\overline{CP}^{n+1} + \phi \overline{CP}^{m+1}$, afterwards to have the Value of CE , that of CP being given, we need only make $UU = CC$, as in the preceeding Case.

SECT. XIV.

THIS being done, let APD (Fig. 3.) represent the Orbit in which the Body makes its Revolution; from the Centre of Force T describe the Circle ASM , and draw the Lines TP , Tp infinitely near each other, and which being produced would meet the Circle at S and s ; erect also the Perpendicular $p m$ upon TP ; and calling TA , r ; TP , x ; $p m$, y ; U the Velocity with which the Body describes the small Arch

Arch Pp ; and supposing that the centripetal Force at A is $= 1$, and that it varies as x^n , we shall have, by known Principles $UU = -x^n \dot{x}$, from whence, by calling c the Height from which the Body ought to fall, in order to acquire the Velocity which it hath at P , we get $\frac{UU}{2} =$

$$\frac{c^{n+1} - x^{n+1}}{n+1}; \text{ but since the Velocity is in a}$$

direct Ratio of the Space run through, and an inverse one of the Time, and the Time is here proportional to the Area described: If m denotes a convenient constant Quantity, it is plain that we

$$\text{shall have the Equation } \frac{c^{n+1} - x^{n+1}}{n+1} =$$

$$\frac{m^2 \times \dot{x}^2 + j^2}{x^2 j^2}, \text{ which, as } x = TA = 1, \text{ will}$$

$$\text{become } \frac{c^{n+1} - 1}{n+1} = m^2; \text{ substituting then}$$

this determined Value of m , we shall find $j =$

$$\frac{\sqrt{c^{n+1} - 1} \times \dot{x}}{\sqrt{c^{n+1} x^2 - x^{n+3} - c^{n+1} + 1}}; \text{ afterwards,}$$

$$\sqrt{c^{n+1} x^2 - x^{n+3} - c^{n+1} + 1}$$

supposing $x = 1 - z$, and neglecting all the Terms in which z arises to more than two Dimensions, the Orbit being by the Hypothesis very little different from a Circle, and making $TP : pm :: TS : Ss$, we shall deduce $Ss =$

$$\frac{1}{1-z} \sqrt{\frac{n+3}{2} \times \frac{n+2}{2} - z^{n+1}}$$

To have the complete Fluent of this Fluxion, we must observe, that when $z = 0$, which happens in the Points of the two Apfides, the Equation $\frac{n+3}{2} \times \frac{n+2}{2} - z^{n+1} = 0$ changes into this, $\frac{n+3}{2} \times \frac{n+2}{2} = 0$;

which gives $z = 0$, and thus in this last Case

or, to abridge it, $z = t$, and thus in this last Case we find that the Fluent of $\frac{1}{1-z} \sqrt{\frac{n+3}{2} \times \frac{n+2}{2} - z^{n+1}}$

is the Angle $\frac{180^\circ}{\sqrt{1-t}} = 180^\circ$, t being an extremely small Quantity; then by drawing the Line

TDN, which passes through the Apfide D, the Arch of the Circle ASN, or the Angle ATD,

will be of $180^\circ \times \sqrt{\frac{n+3}{2} \times \frac{n+2}{2} - z^{n+1}}$

but since by the Hypothesis, the Orbit differs very

very little from a Circle, by the first Case of the Lemma, we shall have $c^{n+1} = \frac{n+3}{2}$, which gives

$$\frac{c^{n+1} - 1}{n+3 \times \frac{n+2}{2} - c^{n+1}} = \frac{1}{n+3}; \text{ and consequently,}$$

the Angle ATD comprehended between the Apfides $\frac{180^\circ}{\sqrt{n+3}}$, as we have found by the first Method.

S E C T. XV.

LET us now suppose that the central Force acts in an inverse Ratio of the Square of the Distance, and that a new Force, which is very small, is added, that shall be to the central Force at A as ϕ to 1, and which encreases in a direct Ratio of the Distance; it is plain that in

this Case we shall have $UU = -\frac{\dot{x}}{x^3} - \phi x \dot{x}$,

and if c denote the Height from which a falling Body, by its Descent to the Point P, will acquire the Velocity which it hath at this Point, it will

become $\frac{UU}{2} = \frac{1}{x} - \frac{1}{c} + \frac{\phi}{2} \times c^2 - x^2$, from

whence we shall form, as in the preceding Case,

the

the Equation $\frac{1}{x} - \frac{1}{c} + \frac{\phi}{2} \times \overline{c^2 - x^2} = \frac{m^2 \times \overline{x^2 + j^2}}{x^2 j^2}$, which, since $x = TA = 1$,

becomes $m^2 = 1 - \frac{1}{c} + \frac{\phi}{2} \times \overline{c^2 - 1}$, and from which we get, $j =$

$$-x \sqrt{1 - \frac{1}{c} + \frac{\phi}{2} \times \overline{c^2 - 1}}$$

$$\sqrt{-1 + \frac{1}{c} - \frac{\phi}{2} \times \overline{c^2 - 1} + x - \frac{x^2}{c} + \frac{\phi}{2} \times \overline{c^2 x^2} - \frac{\phi}{2} x^4}.$$

Then suppose $x = 1 - z$, and rejecting all the Teams in which z has more than two Dimensions, and making $TP : pm :: TS : Ss$, we find $Ss =$

$$\sqrt{\frac{1 - \frac{1}{c} + \frac{\phi}{2} \times \overline{c^2 - 1}}{\frac{1}{c} + 3\phi - \frac{\phi}{2} c^2}} \times \frac{z}{1 - z} \sqrt{\frac{-1 + \frac{2}{c} - \phi c^2 + 2\phi}{\frac{1}{c} + 3\phi - \frac{\phi}{2} c^2}} z - zz.$$

From whence it is plain, that by applying here the same Reasoning as in the preceding Section; the Angle ATD comprehended between the Ap-fides

fides will be of $180^\circ \sqrt{\frac{1 - \frac{1}{c} + \frac{\phi}{2} \times c^2 - 1}{\frac{1}{c} + 3\phi - \frac{\phi}{2} c^2}}$:

But since, by the Hypothesis, the Orbit differs very little from a Circle, we shall have, in the second Case of the Lemma, Section XIII. $n = -2$, $m = 1$, $CP = 1$, and by the Equation

$UU = CC$, we find $1 - \frac{1}{c} + \frac{\phi}{2} \times c^2 - 1 = \frac{1+\phi}{2}$, and $\frac{1}{c} + 3\phi - \frac{\phi}{2} c^2 = \frac{1+4\phi}{2}$, which

gives the Angle ATD of $180^\circ \sqrt{\frac{1+\phi}{1+4\phi}}$. This

Expression of the Angle ATD is the same as that deduced by Sir *Isaac Newton*, from the Hypothesis of a moveable Orbit, (*Princ. Math. Lib. 1. Prop. 45. Coroll. 2.*)

S E C T. XVI.

IF we do not particularly attend to the Series of Calculation in the preceding Section, we shall incline to imagine, that it will agree to the Case of the Moon; but by examining it nearly, it will appear that we have fallen into the Hypo-

E

thesis

thesis of the moveable Orbit: To prove which, we shall make the two following Remarks.

I.

Although the Motion of the Moon is altered by the Action of the Sun (I speak of that Part of the Solar Action which is directed to the Centre of the Earth, and which is that which causes the Change in the Position of the Apfides) yet the Area, described by this Planet around the Earth, is the same, in a given Time, as if this disturbing Force existed not at all, and the Moon made her Revolution by Virtue of the Action of the Earth alone.

II.

Let us suppose that the Circle AM is the *primitive* Orbit of the Moon; that is to say, the Orbit which she describes by Virtue of the Earth's Action alone, and that afterwards the Force of the Sun is joined to it, it is plain that this *foreign* Force has not contributed at all to produce the Velocity with which the Planet described the Circle AM, and that this Velocity is *that* which she has acquired by the Action of the Earth alone, in falling from a Height equal to the Radius TA: The disturbing Force acts not till she is past the Point A, and at this Point the Velocity
which

which it produces becomes nothing; from hence it is evident, that by seeking, as we have done in the preceding Section, the Height c , such as it ought to result from the continued Action of the Sum of the two Forces of which we are speaking, we augment or diminish the real Velocity of the Moon: This is the same Thing as to give a Motion to its Orbit. These two Remarks being observed, it will be easy to apply our second Method to the Case of the Moon.

S E C T. XVII.

WE have then $U\dot{U} = -\frac{\dot{x}}{x^2} - \phi x\dot{x}$,
 and, according to the second Remark,
 $\frac{UU}{2} = \frac{1}{x} - \frac{1}{2} + \frac{\phi}{2} \times \sqrt{1-x^2}$, and conse-
 quently, $\frac{1}{x} - \frac{1}{2} + \frac{\phi}{2} \times \sqrt{1-x^2} = m^2 \times$
 $\frac{\dot{x}^2 + \dot{s}^2}{x^2 \dot{s}^2}$, whence, by making $x = TA = 1$,
 we find $m^2 = \frac{1}{2}$; by putting this Value of m we
 find $\frac{x\dot{s}}{2} = PTp = \frac{-x\dot{x}}{2\sqrt{-1+2x-x^2+\phi x^2-\phi x^4}}$;
 and supposing $x = 1-z$, and rejecting the
 F 2 Terms

Terms in which z has more than two Dimensions, because of the Smallness of the Force ϕ ,

it becomes $PTp = \frac{1}{2\sqrt{1+5\phi}} \times \frac{\overline{1-z} \times \dot{z}}{\sqrt{\frac{2\phi}{1+5\phi}} z - zz}$;

then, calling C the Area of the Semi-circle ASM , and taking the Fluent, we find that the Space APD , comprehended between the two Apfides,

is equal to $\frac{1+4\phi}{1+5\phi^{\frac{3}{2}}} \times C$; from which, ac-

cording to the first Remark, the Time that the Moon employs to describe the Semi-circumference AM , is to the Time T , which she employs to describe the Arch of the Orbit APD , as 1 is to

$\frac{1+4\phi}{1+5\phi^{\frac{3}{2}}}$, or as 1 is to $\frac{1}{\sqrt{1+7\phi}}$, and the

first of these two Times is to the real Time L of the Semi-revolution of the Moon, as $\sqrt{1+\phi}$ to 1; hence $T : L :: \sqrt{1+\phi} : \sqrt{1+7\phi}$; consequently, by substituting the mean Motions in Place of the Times, the Angle described by this Planet in her Revolution from one Apogee to

the other will be found of $360^\circ \sqrt{\frac{1+\phi}{1+7\phi}}$, or

rather of $360^\circ \sqrt{\frac{1-\phi}{1-7\phi}}$, as we have found by the first Method.

S E C T.

S E C T. XVIII.

IT may be objected, that the preceding Methods will be far from giving us the true Quantity of the Motion of the Apogee of the Moon, since nothing but the mean Force of the Sun upon the Moon has been considered, and which has been supposed uniform in all the Points which are equi-distant from the Centre of the Earth, whilst it is demonstrated, that this Force varies according to the Angle of Elongation of the Moon from the Sun, that it is in some Points nothing, in others additive, in others subtractive: But I think it may be demonstrated in the following Manner, that it is indifferent whether the mean Force or the real variable Force be used.

From the Center T (Fig. 4.) which represents the Earth, let the Circle QSM be described, which represents the Orbit of the Moon; let TQ be the Line of the Quadratures; and TS the Line of the Syziges; draw the two Radii TR, Tr, infinitely near each other, and erect the Perpendicular Rm upon TQ; and calling TR, r ; Rm, y ; Rr, z ; we know that the Action of the Sun, that makes the Apogee move, is proportional

portional to $r - \frac{3yy}{r}$, which is nothing, when $y = r \sqrt{\frac{1}{3}}$, which happens at the Point P, the Arch PQ being of $35^\circ 16'$; it is additive along the Arch PQ, and makes the Apogee move retrograde; and it is subtractive along the Arch PS, which is $54^\circ 44'$, and makes the Apogee advance: Now it is plain, that the Force which acts upon the small Arch Rr is expressed by $rs - \frac{3yyj}{r}$; but the Fluent of rs , for the whole Arch PQ, is the Space 2PTQ, and that of $\frac{3yyj}{r}$ is 3PCQ, drawing the Perpendicular PC; then the Sum of the Forces which act upon the Arch PQ, will be expressed by the Area 3PTC - PTQ; it may be found in the same Manner, that the Sum of the Forces which act upon the Arch PS is expressed by the Area 3PTC + PTS, and by taking the Difference of these Sums, we have 3PTC + PTS - 3PTC + PTQ = STQ = SQ $\times \frac{ST}{2}$: Now $\frac{ST}{2}$ represents here the mean Force ϕ ; consequently this mean Force acting uniformly upon the Quadrant of the Circle QS, produces the same Effect as the variable Force; the like Reasoning will hold good for the rest of the Circumference.

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S E C T.

S E C T. XIX.

COROLLARY I.

IT follows, from what has already been said, that the Hypothesis of a moveable Orbit does not give the true Motion of the Apfide, when the central Force is as any one Power of the Distance from the Center; the Results which are deduced from this Hypothesis being then conformable to those which we have found by these two Methods above; but this not happening in the Case when the central Force is compounded of the Sum or of the Difference of more Forces, we must therefore abandon this Hypothesis, and have Recourse to the direct Methods.

S E C T. XX.

COROLLARY II.

IT follows likewise, that when the foreign Force is additive, the Motion of the Apfides is retrograde; and when it is subtractive, the Motion is direct.

It

It must be observed, that with respect to the Moon, only one Part of the foreign Force which acts upon that Planet has been made use of in the preceding Calculations, that is to say, its Gravitation towards the Sun; this is the Part which is directed to the Center of the Earth; the other Part, which acts perpendicularly to the Radius of Revolution, has been neglected, because this Force accelerates the Motion of the Moon in the Quadrant of the Circle QS (Fig. 4.) precisely in the same Proportion that it retards its Motion in the other Quadrant SM; therefore, upon the whole, no Motion in the Apfides can arise from it.

METHOD III.

SECT. XXI.

THIS is a particular Method for the Moon, it supposes that, the Quantity of the Motion of the Apogee for a small Interval of Time, in the *Syziges* and the Quadratures, is known: Now Sir *Isaac Newton* found this Quantity (by what Method is unknown) as may be seen in the Scholium which is at the End of his Theory of the Moon, in the third Book of the *Principia*,

Principia, Ed. I. where he says, “*Similibus computationibus inveni quod Apogæum, ubi in conjunctione vel oppositione solis versatur, progreditur singulis diebus 23' respectu fixarum; ubi vero in quadraturis est, regreditur singulis diebus 16' $\frac{1}{2}$ circiter.*”

This being supposed, it is necessary to demonstrate the following Theorem.

T H E O R E M.

Suppose the Motion of a Body that makes its Revolution in a Circle is compounded of two particular Motions, one of which is uniform, the other variable, if the Difference of the Velocity in the different Points of the Circumference from the greatest Velocity, be as the Square of the Sine of the Angle comprehended between each of the Points and the Point of the greatest Velocity, an ellipsis may be found whose Area will represent the mean Motion.

D E M O N S T R A T I O N.

From the Center T describe the Circle QSM, (Fig. 5.) in which a Body may be conceived to make its Revolution with a Motion compounded of two Motions, the one uniform, and the other variable; drawing the two Radii TQ, TS,

G perpen-

38 The THEORY of

perpendicular to each other, let Q be the Point of the greatest Velocity, and calling e the small horary arch described at Q by the Power of the uniform Motion, n that described at the same Time by the variable Motion; we shall have $e + n$ for the Arch Qq , described by the greatest Velocity; and if m denote the small horary Arch described at S , by the variable Motion, by taking it negative, we shall have $e - m$ for the Arch described by that Velocity, which is supposed the least: The Difference of these Velocities, namely, $m + n$, must always be as the Square of the Sine of the Angle comprehended between the Point Q and any other Point L , taken in the Circumference; if we draw the Sine LR , and the Radius LT , this Difference at the Point L will be $\frac{m + n}{\text{LT}} \times \frac{LR^2}{\text{LT}^3}$, and consequently the total horary Motion in this Point will be $e + n - \frac{m + n}{\text{LT}} \times \frac{LR^2}{\text{LT}^3}$.

Now let us conceive a Curve HAC such, that the Areas which are there described by the Radius of Revolution may be proportional to the Time, and B being the Point where it cuts the Radius TL , draw BG parallel to LR , and calling BG , u ; TB , x ; TH , b ; TQ , r ; and from the Center T describing the Arch Hr ;
we

we shall have $HS = \frac{b}{r} \times \overline{e+n}$, and the Area
 $HTS = \frac{bb}{2r} \times \overline{e+n}$; in like manner as the
 circular Motion at the Point L was $\overline{e+n} - \overline{m+n}$

$\times \frac{\overline{LR}^2}{\overline{TL}}$, having $TL : LR :: TB : BG$; this

Motion reduced to the Point B, will be $\frac{x}{r} \times$

$\overline{e+n} - \frac{uu}{rx} \times \overline{m+n}$; and the Area described

will be $\frac{xx}{2r} \times \overline{e+n} - \frac{uu}{2r} \times \overline{m+n}$; and as these

Areas must be equal, we shall have $\frac{xx}{2r} \times \overline{e+n}$

$- \frac{uu}{2r} \times \overline{m+n} = \frac{bb}{2r} \times \overline{e+n}$; from whence

we draw $xx - \frac{m+n}{e+n} uu = bb$. This proves that

the Curve HAC is an Ellipsis whose Semi-axis
 TH, b , is to the Semi-axis TA, which call a ,

as $\sqrt{e-m}$ is to $\sqrt{e+n}$; that is to say, recipro-

cally as the Square-roots of the Velocities at the

Points Q and S; the Area then of this Ellipsis

will represent the mean uniform Motion. Q.E.D.

If the Excess of Velocity in all the Points L
 above the smallest Velocity at the Point S, increases

as the Square of the Sine of the Angle LTS, one might also demonstrate, in the same Manner, that the Curve HAC is an Ellipsis.

S E C T. XXII.

MR. *Macbin*, having contented himself with simply giving us this Theorem *, I have been willing to give its Demonstration, both on account of its Beauty, and also for the great Facility it affords, not only in determining the Motion of the Apogee, but likewise a very great Number of other Circumstances of the Moon's Motion, as the learned Author has shewn in the Treatise just now quoted.

Suppose then that TQ denotes the Line of the Quadratures, and TS the Line of the *Syziges*, and the Motion in the Circle QSM is the Motion of the Moon, according to her Elongation from her Apogee, that is to say, that this Motion is the Sum or the Difference of the uniform Motion of the Moon and the Motion of the Apogee; now as this last is proportional to the Force which produces it, and this Force is proportional to the

* The Laws of the Moon's Motion, P. 18.

Quantity

Quantity $r - \frac{3yy}{r}$, by calling LR, y , we see

that this Force is expressed by r at the Point Q; and so the Difference between the Velocity of circular Motion at the Point Q, and the Velocity of that Motion at the Point L, is proportional to

$\frac{3yy}{r}$, that is to say, to the Square of the Sine

of the Angle of Elongation at the Point Q; then because it is demonstrated that the Curve HAC will be an Ellipsis, and its area will represent the mean Motion of the Moon, whilst the Area of the Circle will represent the Motion of that Planet from her Apogee; the Points P, P, where this Ellipsis cuts the Circle, are the Points

in which we have $r - \frac{3yy}{r} = 0$, or $yy = \frac{rr}{3}$;

then by the Property of the Ellipsis we shall have

in these Points, $rr - \frac{rr}{3} = bb - \frac{bb}{aa} \times \frac{rr}{3}$,

which gives $rr = \frac{3aabb}{2aa + bb}$; but since the Area

of an Ellipsis is to the Area of a Circle, as ab to rr ,

that is to say, as 1 to $\frac{3ab}{2aa + bb}$, we may deduce

this Proportion; the mean Motion of the Moon is to the mean Motion of the Moon from her

Apogee, as 1 is to $\frac{3ab}{2aa + bb}$; then if we suppose

that

that the Elongation of the Moon from her Apogee is of 360 degrees, that is to say, she makes her Revolution from one Apogee to the other,

we shall have $360^\circ \times \frac{2aa + bb}{3ab} = 360^\circ \times$

$\frac{3e + 2n - m}{3\sqrt{e + n} \times e - m}$ for the mean Motion, but

we have $e = 13^\circ 10' 35''$, $n = 16' 20''$ and $m = 23'$, and by the mean of these Values we find that the mean Motion is $363^\circ 6' \frac{1}{2}$, and consequently the Progression of the Apogee $3^\circ 6' \frac{1}{2}$: The small Difference between this Result and that of the two preceding Methods will vanish, if we consider that the Motion of the Apogee in the Time of the Syziges and Quadratures has not been exactly determined by Sir Isaac Newton, as he himself acquaints us; for says he, *Computationes autem ut nimis perplexas & approximationibus impeditas, neque satis accuratas, apponere non lubet.* If we determine the Values of m and n by the Formulæ which the Hypothesis of a moveable Ellipsis furnishes, we shall only find about $1^\circ 37' \frac{1}{2}$ for the Motion of the Apogee.

12. III 54

The END.

ERRATA.

Page 5	Line 2	<i>medius annuus</i>
22	16	from the Center
	21	Velocity acquired